Correlator expansion approach to stationary states of weakly coupled cavity arrays

Elena del Valle and Michael J. Hartmann

Physik Department, Technische Universität München, James-Franck-Straße, 85748 Garching, Germany

Abstract. We introduce a method for calculating the stationary state of a translation invariant array of weakly coupled cavities in the presence of dissipation and coherent as well as incoherent drives. Instead of computing the full density matrix our method directly calculates the correlation functions which are relevant for obtaining all local quantities of interest. It considers an expansion of the correlation functions and their equations of motion in powers of the photon tunneling rate between adjacent cavities, leading to an exact second order solution for any number of cavities. Our method provides a controllable approximation for weak tunneling rates applicable to the strongly correlated regime that is dominated by nonlinearities in the cavities and thus of high interest.

1. Introduction

The study of light matter interactions that are enhanced by confining light fields in electromagnetic cavities has been a thriving discipline of Quantum Optics throughout the last decades. Particularly with the advent of realizations of the so called strong coupling regime where the strength of the interaction between a photon and a quantum emitter exceeds the decay mechanisms for photons and emitters, experimental investigations of the coherent interactions between single emitters and individual photons became possible [1].

In recent years, a new direction of cavity quantum electrodynamics (cavity-QED) has developed, in which multiple cavities that are coupled via the exchange of photons are considered. Such setups are particularly intriguing if the cavities are connected forming an array and the strong coupling regime is achieved in each cavity of the array. These devices would then give rise to quantum many-body systems of strongly interacting photons and polaritons [3, 4, 5]. As an alternative to a cavity array, one may also consider optical fibers that couple to nearby atoms [6, 7] or even clouds of Rydberg atoms that are optically thick in free space [8]. Both these systems avoid the need to build mutually resonant cavities, which is possible [9] but can be rather challenging in the optical range. For microwave photons it is however perfectly feasible to build large arrays of mutually resonant cavities on one chip in an architecture known as circuit-QED [10, 11].

For strongly interacting polaritons and photons in coupled arrays of micro-cavities and optical fibers, possibilities to observe equilibrium phenomena, such as a Mott insulator [3, 4, 5] or a Tonks-Girardeau gas [6, 7], have mostly been addressed so far and the development has been summarized in the reviews [12, 13, 10]. In every experiment that involves light-matter interactions, some photons will however inevitably be lost from the structure due to imperfect light confinement or emitter relaxation so that thermal states are no longer an appropriate description of the system. To compensate for such losses, coupled cavity arrays are thus most naturally studied in a regime where a coherent or incoherent input continuously replaces the dissipated excitations. This mode of operation eventually gives rise to a driven dissipative regime, where the dynamical balance of loading and loss processes leads to the emergence of stationary states. Yet the properties of stationary states of driven dissipative systems are only explored to a much lesser degree than the properties of thermal equilibrium states. For coupled cavities, small arrays have been considered exactly [14] and mean field approaches for larger arrays have been employed [15]. Moreover numerical studies found signatures for crystallization [16] and photon solids were predicted for arrays with cross Kerr interactions [17].

As driven dissipative quantum many-body systems have to date only barely been explored, there is a need for technical tools for their efficient description. Here we introduce a perturbative technique for the calculation of the physical properties of stationary states of driven dissipative cavity arrays. Our approach assumes a large,

translation invariant cavity array where the quantum states of all cavities are identical. Instead of computing the full density matrix, it directly calculates the correlation functions which are relevant for obtaining all local quantities of interest. A power expansion of the formal exact solution to second order in the photon tunneling rate between adjacent cavities, provides semi-analytical results for any number of cavities.

2. Exact solution in the steady state

We consider an array of cavities that are coupled via mutual photon tunneling and where each cavity is doped with a Kerr nonlinear medium that generates a strong photon-photon interactions. In a frame that rotates at the frequency of the coherent drive lasers, this system is described by a Bose-Hubbard Hamiltonian with local coherent drives ($\hbar = 1$),

$$H_N = \sum_{i=1}^{N} \left[\Delta a_i^\dagger a_i + \frac{U}{2} a_i^\dagger a_i^\dagger a_i a_i + \Omega(a_i^\dagger + a_i) \right] + J \sum_{i=1}^{N-1} (a_i^\dagger a_{i+1} + a_{i+1}^\dagger a_i) + J(a_N^\dagger a_1 + a_1^\dagger a_N) \tag{1}$$

where we assumed periodic boundary conditions so that N cavities form a circle coupling to nearest left and right neighbors. Here, $\Delta = \omega_a - \omega_{\rm L}$ is the detuning of the cavity resonance frequency ω_a from the laser frequency $\omega_{\rm L}$, U is the strength of the Kerr nonlinearity, J the rate of photon tunneling between the cavities and Ω the drive amplitude of a coherent laser drive. We consider the local energy scales orders of magnitude smaller than the cavity frequency, that is, U, J, Δ, $\Omega \ll \omega_a$, so that the rotating wave approximation can be applied. Adding decay and incoherent pumping of the modes, the total Liouvillian that describes the dynamics of this system reads

$$\partial_t \tilde{\rho} = i[\tilde{\rho}, H_N] + \frac{\gamma}{2} \sum_{i=1}^{N} (2a_i \tilde{\rho} a_i^\dagger - a_i^\dagger a_i \tilde{\rho} - \tilde{\rho} a_i^\dagger a_i) + \frac{P}{2} \sum_{i=1}^{N} (2a_i^\dagger \tilde{\rho} a_i - a_i a_i^\dagger \tilde{\rho} - \tilde{\rho} a_i a_i^\dagger), \tag{2}$$

where $\tilde{\rho}$ denotes the total density matrix of the cavity array. γ is the rate of photon decay and P the rate at which photons are incoherently pumped into the device. We are interested in the steady state under a continuous excitation of each of the cavities, represented by the reduced density matrix ρ of a single cavity. As all cavities have exactly the same properties and dynamics, we find $\rho_1 = \rho_2 = \ldots = \rho_N = \rho$, where ρ_j is the reduced density matrix of cavity number j.

Here, instead of obtaining the full density matrix of the array in the steady state ($\partial_t \tilde{\rho} = 0$) and then tracing out all cavities but one, to obtain ρ, we compute the steady state properties directly in the form of the local mean values of all possible operators defining the system. These can be written as

$$\langle a_i^{\dagger m} a_i^n \rangle = {\rm Tr}(\tilde{\rho} a_i^{\dagger m} a_i^n) \tag{3}$$

where m and n are integers. Let us call $\mathcal{O}$ the set of operators the averages of which correspond to the correlators required to describe the full N-cavity system, i.e., $\mathcal{O}$ includes all the sought observables as well as operators which couple to them through the equations of motion, $\langle a_1^{\dagger m} a_1^n a_2^{\dagger \mu} a_2^\nu \dots a_N^{\dagger \alpha} a_N^\beta \rangle$. To simplify notation, we express this general correlator as $\{\{m,n\},\{\mu,\nu\}\dots\{\alpha,\beta\}\}$. Naturally, in the case of an anharmonic mode or in the presence of any anharmonicity, one must choose an appropriate truncation in the number of excitations, i.e. $n, m \leq n_{\max}$, that also truncates the number of correlations in this set. For a given driving intensity, the truncation must be high enough to yield converged, accurate results.

From the master equation (2) one can obtain the set of coupled equations for the full set of operators $\mathcal{O}$, which we write in the matrix form,

$$\partial_t \tilde{v} = \tilde{M}\tilde{v} + \tilde{I}\,, \tag{4}$$

with all correlators forming the vector $\tilde{v}$, i.e. $\tilde{v}^{\mathrm{T}} = (\langle a_1\rangle, \langle a_1^\dagger\rangle, \dots, \langle a_1 a_2\rangle, \langle a_1^\dagger a_2^2\rangle \dots)$, where the exponent T denotes the transpose. The coefficient matrix $\tilde{M}$ and vector $\tilde{I}$ are derived from the master equation in a systematic way [18, 19], as we explicitly show in the Appendix. The solution of Eq. (4) is completely equivalent to computing the correlators as in Eq. (3), from the density matrix obtained by solving Eq. (2). The exact solution in the steady state (if it is unique and exists) simply reads

$$\tilde{v} = -\tilde{M}^{-1}\tilde{I} \tag{5}$$

We can considerably reduce the number of operators to a minimal set by making use of the translational symmetry in the 1D chain (circle). That is, all correlators that are left or right circular rotations of the elements in $\{\{m,n\},\{\mu,\nu\}\dots\{\alpha,\beta\}\}$, such as $\{\{\alpha,\beta\},\{m,n\},\{\mu,\nu\}\dots\}$ or $\{\{\alpha,\beta\},\dots,\{\mu,\nu\},\{m,n\}\}$, are redundant because they are exactly the same. There is, therefore, a maximum of $2N$ representations of the same correlator (less if some of the pairs are $\{0,0\}$ or mutually equal). We can choose an arbitrary rule to systematically keep only one representative of such set of redundant correlators, for instance, we choose the ones where:

(i) The nonzero sets are always to the left and as cluttered together as possible, such as $\{\{4,2\},\{3,0\},\{0,0\}\dots\}$

(ii) The largest sum of indexes is most to the left: $m+n \geq \mu+\nu \geq \dots \geq \alpha+\beta$, such as $\{\{3,3\},\{1,3\},\{0,2\},\{1,0\}\}$. Together with the previous rule, this may give things like $\{\{1,3\},\{3,3\},\{1,0\},\{0,0\},\dots\}$.

(iii) If there are two pairs with an equal sum, the one with the largest first index is left-most $m \geq \mu$. Together with the previous rule, this may give things like $\{\{4,2\},\{3,3\},\{0,0\},\dots\}$.

With this, the vectorial Hilbert space is significantly reduced, for instance from 6559 to 1033 for $N = 4$ and $n_{\max} = 2$. From $\tilde{v}$ we thus extract a new vector v that only contains the minimal set of correlators. The equation of motion for v is then written in terms of a new matrix, M, reconstructed by removing the redundant rows and summing the coefficients of the redundant columns of $\tilde{M}$, and a new vector, I removing the redundant

rows of $\tilde{I}$. The exact stationary state solution of the reduced system of equations is now given by

$$v = -M^{-1} I \,. \tag{6}$$

2.1. One cavity, $N = 1$, and the uncoupled limit

In the uncoupled limit, $J = 0$, the calculation can be reduced to a single cavity. The operators for each cavity, $\langle a_i^{\dagger m} a_i^n \rangle$, are the same for all $i = 1, \ldots, N$, so let us denote them by a common name, $\langle a^{\dagger m} a^n \rangle = \{\{m, n\}\}$, without any index. We can define the vector v_a of all possible individual cavity operators,

$$v_a^{\mathrm{T}} = \big(\langle a \rangle, \langle a^\dagger \rangle, \langle a^\dagger a \rangle, \ldots\big) \,. \tag{7}$$

In the case of one cavity or many which are uncoupled, the full ensemble vector v reduces trivially to $v = v_a$. The equation of motion for this system reads $\partial_t v_a = M_a v_a + I_a$ and its stationary solution is,

$$v_a = -M_a^{-1} I_a \,. \tag{8}$$

2.2. $N \leq 4$ cavities

Four is the minimum number of identical cavities needed to obtain a general solution to second order in J that is valid for any N. The reason is that four coupled systems imply a qualitative change, as compared to two or three, as each of them is no longer in contact with all the others. This represents the general case to second order in J.

In this case, the vector of correlators v not only contains the subset v_a with $\{\{m, n\}, \{0, 0\}, \{0, 0\}, \{0, 0\}\}$, but also another four new subsets that include cross correlations with two, three or four cavities. The first one, which we call v_b, includes correlators with two cavities, $\langle a_1^{\dagger m} a_1^n a_2^{\dagger \mu} a_2^\nu \rangle = \langle a_2^{\dagger m} a_2^n a_1^{\dagger \mu} a_1^\nu \rangle = \ldots = \langle a^{\dagger m} a^n b^{\dagger \mu} b^\nu \rangle = \{\{m, n\}, \{\mu, \nu\}, \{0, 0\}, \{0, 0\}\}$. We denote with b the photon annihilation operator for the second cavity, and apply the rules to extract the minimal set of operators ($m + n \geq \mu + \nu$, with $m \geq \mu$ in case of degeneracy of the sum). For example, for a truncation in each cavity system with $n_{\max} = 2$ photons, we thus have a dimension of 8 for v_a and 36 for v_b. A third subset, v_c, includes correlators with three cavities, $\langle a_1^{\dagger m} a_1^n a_2^{\dagger \mu} a_2^\nu a_3^{\dagger p} a_3^q \rangle = \langle a_2^{\dagger m} a_2^n a_1^{\dagger \mu} a_1^\nu a_3^{\dagger p} a_3^q \rangle = \ldots = \langle a^{\dagger m} a^n b^{\dagger \mu} b^\nu c^{\dagger p} c^q \rangle = \{\{m, n\}, \{\mu, \nu\}, \{p, q\}, \{0, 0\}\}$. Similarly to b, we denote with c the photon annihilation operator in the third cavity, where we have applied the circular rules to obtain the minimal set of operators. Finally, specifically to $N = 4$, we need to consider v_d, which includes correlators with 4 cavities $\langle a_1^{\dagger m} a_1^n a_2^{\dagger \mu} a_2^\nu a_3^{\dagger p} a_3^q a_4^{\dagger s} a_4^t \rangle = \ldots = \langle a^{\dagger m} a^n b^{\dagger \mu} b^\nu c^{\dagger p} c^q d^{\dagger s} d^t \rangle = \{\{m, n\}, \{\mu, \nu\}, \{p, q\}, \{t, s\}\}$, d is the photon annihilation operator in the fourth cavity, and v_e which includes operators of two cavities at alternate positions, $\langle a_1^{\dagger m} a_1^n a_3^{\dagger p} a_3^q \rangle = \langle a_2^{\dagger m} a_2^n a_4^{\dagger p} a_4^q \rangle = \ldots = \langle a^{\dagger m} a^n c^{\dagger p} c^q \rangle = \{\{m, n\}, \{0, 0\}, \{p, q\}, \{0, 0\}\}$.

We can rewrite Eq. (4) in terms of these subsets of correlators, each with a different dimension, as five coupled matrix equations,

$$\partial_t v_a = (M_a + iJS_a) v_a + I_a + iJR_{ab} v_b \,, \tag{9}$$

$$\partial_t v_b = (M_b + iJS_b)v_b + (B_{ba} + iJR_{ba})v_a + iJR_{bc}v_c\,, \tag{10}$$
$$\partial_t v_c = (M_c + iJS_c)v_c + (B_{cb} + iJR_{cb})v_b + iJR_{cd}v_d + (B_{ce} + iJR_{ce})v_e\,, \tag{11}$$
$$\partial_t v_d = (M_d + iJS_d)v_d + (B_{dc} + iJR_{dc})v_c\,, \tag{12}$$
$$\partial_t v_e = (M_e + iJS_e)v_e + (B_{ea} + iJR_{ea})v_a + iJR_{ec}v_c\,. \tag{13}$$

Here, we have separated the effect of the hopping J into the self-renormalization matrices S, and the linking matrices R, that only contain integer numbers. The vector I_a and matrices B, contain only the driving parameters Ω and P (coherent or incoherent). Other internal parameters such as Δ, γ and U enter in the matrices M.

One can solve these equations recurrently in the steady state, from bottom to top. This may be useful for a small number of cavities where the expressions are simple. For instance, for $N = 2$, the system reduces to v_a and v_b with Eqs. (9)–(10), and solutions:

$$v_a = -\left(M_a + iJS_a + iJR_{ab}F_{ba}\right)^{-1} I_a\,, \tag{14}$$
$$v_b = F_{ba}v_a\,, \tag{15}$$

where $F_{ba} = -\left(M_b + iJS_b\right)^{-1}(B_{ba} + iJR_{ba})v_a$. Similarly, for $N = 3$ we have:

$$v_a = -\left(M_a + iJS_a + iJR_{ab}F_{ba}\right)^{-1} I_a\,, \tag{16}$$
$$v_b = F_{ba}v_a\,, \quad \text{and} \quad v_c = F_{cb}v_b\,, \tag{17}$$

where $F_{ba} = -\left(M_b + iJS_b + iJR_{bc}F_{cb}\right)^{-1}(B_{ba}+iJR_{ba})$ and $F_{cb} = -\left(M_c + iJS_c\right)^{-1}(B_{cb}+iJR_{cb})$. This recursive procedure is possible in principle for any N, although, in general not very practical, given that the exact solution can also be obtained by simply inverting one matrix, M, as in Eq. (6). Anyhow, obtaining the exact solution becomes exceedingly cumbersome for a large number of cavities, $N \gg 1$. In the following we therefore concentrate on finding an approximate solution.

3. Approximated solution to second order in J

In order to find an approximate semi-analytical expression for the steady state of a cavity, v_a, we expand both the correlators in v and the set of equations (9)–(13) in powers of J up to second order. More precisely, second order for v_a, requires for consistency the following lower orders in the other subsets:

$$v_a = v_a^{(0)} + Jv_a^{(1)} + J^2 v_a^{(2)} + \dots\,, \tag{18}$$
$$v_b = v_b^{(0)} + Jv_b^{(1)} + \dots\,, \tag{19}$$
$$v_c = v_c^{(0)} + \dots\,, \tag{20}$$
$$v_d = v_d^{(0)} + \dots\,, \tag{21}$$
$$v_e = v_e^{(0)} + \dots\,. \tag{22}$$

The expanded equations read in these terms:

$$\partial_t v_a = 0 = \left[M_a v_a^{(0)} + I_a\right] + J\left[M_a v_a^{(1)} + iS_a v_a^{(0)} + iR_{ab}v_b^{(0)}\right] + J^2\left[M_a v_a^{(2)} + iS_a v_a^{(1)} + iR_{ab}v_b^{(1)}\right] + \dots\,, \tag{23}$$

$$\partial_t v_b = 0 = \Big[M_b v_b^{(0)} + B_{ba} v_a^{(0)}\Big] + J\Big[M_b v_b^{(1)} + iS_b v_b^{(0)} + B_{ba} v_a^{(1)} + iR_{ba} v_a^{(0)} + iR_{bc} v_c^{(0)}\Big] + \dots . \quad (24)$$

We truncate the equations at this point since the solutions obtained by setting each square bracket to zero,

$$v_a^{(0)} = -M_a^{-1} I_a\,, \quad v_b^{(0)} = -M_b^{-1} B_{ba} v_a^{(0)}\,, \quad (25)$$

$$v_a^{(1)} = -M_a^{-1}\Big[iS_a v_a^{(0)} + iR_{ab} v_b^{(0)}\Big]\,, \quad (26)$$

$$v_b^{(1)} = -M_b^{-1}\Big[iR_{ba} v_a^{(0)} + B_{ba} v_a^{(1)} + iS_b v_b^{(0)} + iR_{bc} v_c^{(0)}\Big]\,, \quad (27)$$

$$v_a^{(2)} = -M_a^{-1}\Big[iS_a v_a^{(1)} + iR_{ab} v_b^{(1)}\Big]\,, \quad (28)$$

ultimately depend on $v_c^{(0)}$ only. Obtaining $v_c^{(0)}$ from the equations would in turn require the knowledge of $v_e^{(0)}$ but this is not needed given that the zero order is simply the uncoupled limit, that is, products of the solutions for $N = 1$ as in Eq. (8). For instance, the uncorrelated solution for $v_b^{(0)}$, corresponding to $\langle a^{\dagger m} a^n b^{\dagger \mu} b^\nu\rangle^{(0)} = \langle a^{\dagger m} a^n\rangle\langle a^{\dagger \mu} a^\nu\rangle$, can be directly obtained through the product of twice $v_a^{(0)}$, as $v_b^{(0)} = v_a^{(0)} X_b v_a^{(0)}$, where X_b is the corresponding mixing matrix obtained by inspection. This is completely equivalent to the linear algebra solution of Eq. (25). The same applies for $v_c^{(0)}$ but with two mixing matrices, $v_c^{(0)} = v_a^{(0)} X_{c1} v_a^{(0)} X_{c2} v_a^{(0)}$.

These solutions are valid for $N \geq 4$, since adding more cavities to the circle does not produce any structural qualitative change to second order in J. The approximation is better the larger the N. It is formally the same for $N = 2$ and 3 (setting $v_c^{(0)} = 0$ for $N = 2$), but differs quantitatively to first and second order, respectively due to different coefficients in the equations.

4. Comparison between the exact and approximated results

Since the approximated solutions are single valued, this method cannot reproduce regimes where several steady states are compatible for the individual system (corresponding to different steady states of the ensemble) or any other instability regions like lasing. Its perturbative nature allows it only to describe regimes where the coupling is smaller than the effective decoherence or driving. More precisely, the *weak coupling regime*, where new collective eigen-modes are not required to describe the ensemble dynamics.

We illustrate the interest of this method by comparing in Fig. 1 the exact solution for $N = 4$ (in solid black) with the approximated ones, valid for $N \geq 4$. We have chosen two quantities of interest, the mean cavity population, $n_a = \langle a^\dagger a\rangle$, and its second order coherence function at zero delay, $g^{(2)}(0) = \langle a^\dagger a^\dagger a a\rangle / n_a^2$. Figs. 1(a) and (b) show that both are well approximated by the second order solution (in dashed red) as long as $J < \Omega, \gamma$. Lower order approximations (in blue and green) deviate from the exact solution at even lower J. In Figs. 1(c) and (d), we fix $J = 0.3\gamma < \Omega = 0.5\gamma$, and scan

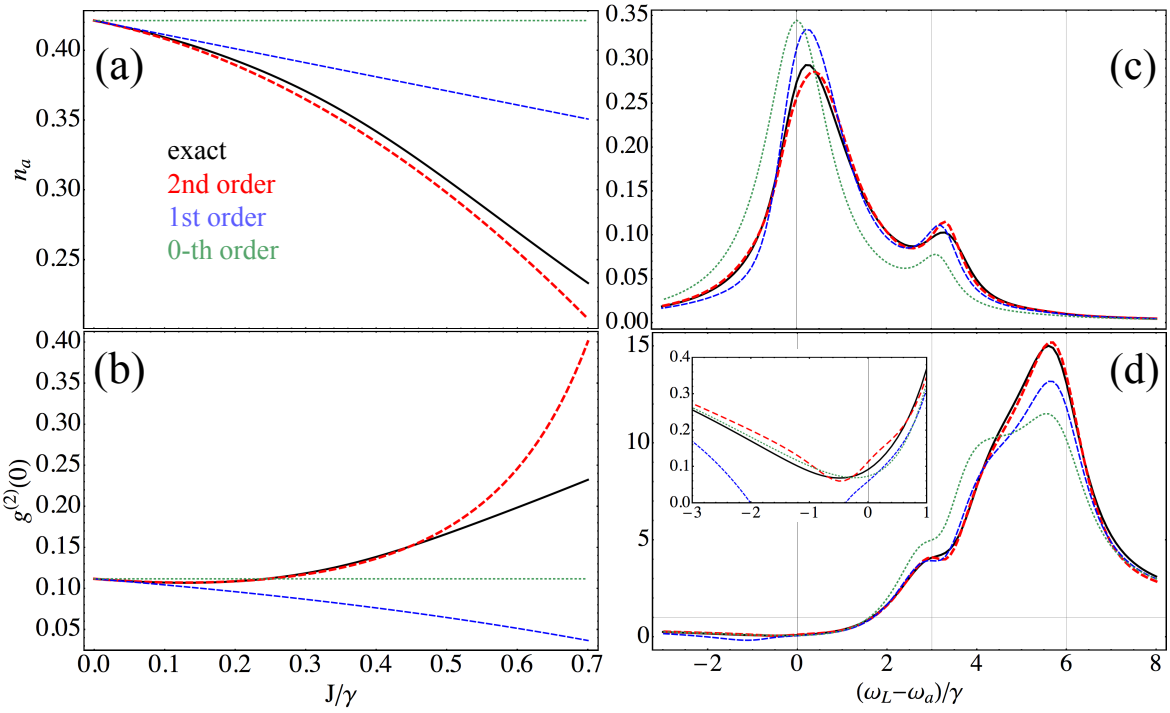

Figure 1. Mean cavity population, n_a, and second order coherence function, $g^{(2)}(0)$. The exact solution for $N = 4$ cavities (solid black) is compared to second (dashed red), first (dashed blue) and zero (dotted green) order approximations, valid for $N \geq 4$. In (a) and (b) we fix $\Omega = 0.7\gamma$ and $\Delta = 0$ and vary the photon tunneling rate J. In (c) and (d) we fix $\Omega = 0.5\gamma$ and $J = 0.3\gamma$ and vary the laser frequency $\omega_{\rm L}$. In the inset of (d), we have magnified the region around 0. Other parameters: $U = 6\gamma$, $P = 0$, $n_{\rm max} = 2$.

the system resonances by tuning the laser frequency. In this case, we observe that the second order approximation remains very close to the exact solution for all frequencies while the first and zeroth order deviate from it, notably, close to the various cavity resonances at 0, $U/2$, U (marked with vertical lines). The first order approximation breaks down near the one-photon resonance at 0 as evidenced by the negative value of $g^{(2)}(0)$ in the zoom-inset of Fig. 1(d).

Due to the form of the dissipation terms in Eq. (2) we cannot illustrate the method in the absence of a coherent drive, only under the action of an incoherent pumping. An incoherent pump bringing the system into a steady state is equivalent to letting each cavity interact with a thermal bath, where $P = n_{\rm T}\gamma_0$ and $\gamma = (1+n_{\rm T})\gamma_0$. Here γ_0 is the decay rate at zero temperature and $n_{\rm T}$ the occupation number of the bath at temperature T, which are identical for all cavities in our considerations. Hence for the case of purely dissipative dynamics with $H_N = 0$ in Eq. (2), the steady state $\tilde{\rho}_{\rm th}$ is a product of thermal states at temperature T for each cavity, that is, $\tilde{\rho}_{\rm th} = Z^{-1}\sum_n e^{-\omega_a n/(k_B T)}$, where k_B is Boltzmann constant, n the total number of excitations in the system and Z the partition sum. For $\Omega = 0$ the Hamiltonian in Eq. (1) conserves the number of excitations in the system and thus $[H_N, \tilde{\rho}_{\rm th}] = 0$ so that $\tilde{\rho}_{\rm th}$ is the steady state

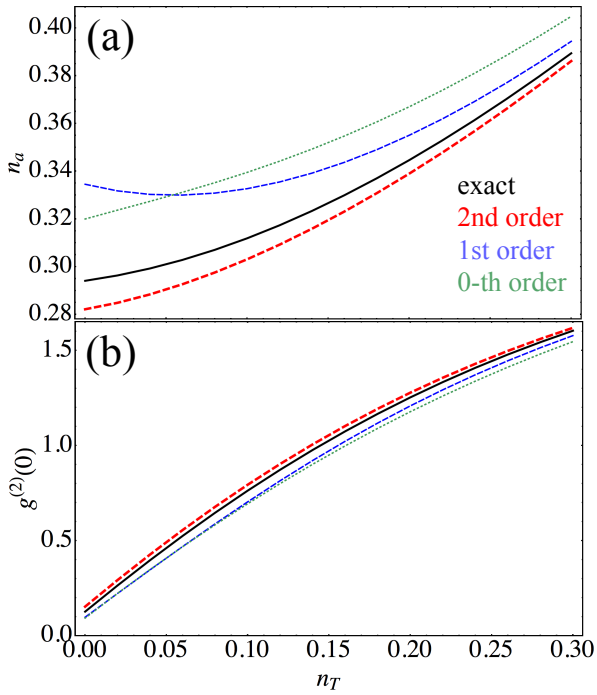

Figure 2. (a) Mean cavity population, n_a, and (b) second order coherence function, $g^{(2)}(0)$, as a function of the thermal bath occupation. The exact solution for $N = 4$ cavities (solid black) is compared to second (dashed red), first (dashed blue) and zero (dotted green) order approximations, valid for $N \geq 4$. Parameters are chosen to maximize the cavity population in Fig. 1(c): $U = 6\gamma_0$, $\Omega = 0.5\gamma_0$, $J = 0.3\gamma_0$, $\omega_{\rm L} = \omega_a + 0.25\gamma_0$ and $n_{\rm max} = 2$.

even in the presence of unitary dynamics generated by H_N, independently of the value of J. Of course the rotating wave approximation that has been applied to derive the Hamiltonian (1) is only valid for U, $J \ll \omega_a$. In this regime a thermal bath as described by the dissipation terms in Eq. (2) is "blind" to the energy scales U, J and all cavities will eventually be in thermal equilibrium with their bath, $n_a = n_{\rm T}$ and $g^{(2)}(0) = 2$, regardless of the hopping J and therefore other neighboring cavities. The dynamics and spectrum of emission (out of the scope of the present study) do, however, depend on the microscopic properties of the cavities. For example, larger J and γ_0 would accelerate the thermalization of the cavity array.

If, on the other hand, a coherent and an incoherent drive are both present, the steady state becomes nontrivial. Moreover, our approach is suited for exploring this experimentally relevant scenario that describes coherently driven cavities in the presence of thermal background radiation. Fig. 2 shows n_a and $g^{(2)}(0)$ as a function of the thermal bath occupation number $n_{\rm T}$, which increases with increasing temperature, for the case of maximum cavity population in Fig. 1(c). Both the exact and the approximated solutions converge at high temperature to the thermalized steady state ($n_a = n_{\rm T}$ and $g^{(2)}(0) = 2$). At low temperatures, where the emission is antibunched and each cavity behaves like a single-photon emitter, first and zero order approximations differ strongly

from the exact solution, especially when computing n_a, while, again, the second order approximation follows it quite smoothly.

It is interesting to note that the steady state cavity spectrum of emission could also be obtained from this approach without recurring to the quantum regression theorem and, therefore, to deriving any time dynamics. The alternative to such complications is to look into the steady state occupation, as a function of its natural frequency, of a mode, that is weakly coupled to the cavity array and which plays the role of the detector. We showed with coworkers the equivalence between this quantity and the power spectrum [19].

5. Conclusions

We have presented a method to solve for the steady state of coupled cavities in a circular 1D array with translational symmetry, to second order in the photon tunneling rate, J. This method can be generalized to any set of identical weakly coupled systems, being in a 1D, 2D or 3D arrangement.

We consider any type of driving of the cavities (coherent or incoherent), dissipation and nonlinearities. We first derive the equations of motion for a minimal set of relevant correlators, v_a, and then perform a power expansion of both the equations and the solutions to obtain semi-analytical expressions for $v_a \approx v_a^{(0)} + Jv_a^{(1)} + J^2 v_a^{(2)}$. The approximated solution is invariant for $N \geq 4$ cavities due to the nearest neighbor nature of the coupling. We have finally illustrated the performance of our method with an example of four weakly coupled cavities under a coherent drive and temperature.

6. Acknowledgements

We acknowledge fruitful discussions with Peter Degenfeld-Schonburg. EdV acknowledges support from the Alexander von Humboldt foundation and MJH from the German Research Foundation (DFG) via the Emmy Noether project HA 5593/1-1 and the CRC 631.

Appendix: Equations for the correlators

In this appendix we provide the matrix $\tilde{M}$ and vector $\tilde{I}$ appearing in Eq. (4), which are the starting point of the described procedure. Let us consider operators for only two adjacent cavities, $\langle a_1^{\dagger m} a_1^n a_2^{\dagger \mu} a_2^\nu \rangle$, as the general case of an array is an straight-forward generalization. Then, we can obtain their equations of motion from the full master equation as

$$\partial_t \langle a_1^{\dagger m} a_1^n a_2^{\dagger \mu} a_2^\nu \rangle = \mathrm{Tr}(\partial_t \rho \, a_1^{\dagger m} a_1^n a_2^{\dagger \mu} a_2^\nu) = \sum_{k,p,\alpha,\beta} \tilde{R}_{\substack{m,n,\mu,\nu \\ k,p,\alpha,\beta}} \langle a_1^{\dagger k} a_1^p a_2^{\dagger \alpha} a_2^\beta \rangle \, . \quad (29)$$

The corresponding elements in $\tilde{R}$ are given by [18]:

$$\tilde{R}_{\substack{m,n,\mu,\nu\\m,n,\mu,\nu}} = i\Delta(m-n) - \frac{\gamma}{2}(m+n) + i\frac{U}{2}[m(m-1) - n(n-1)] + i\Delta(\mu-\nu) - \frac{\gamma}{2}(\mu+\nu) + i\frac{U}{2}[\mu(\mu-1) - \nu(\nu-1)]\,, \tag{30}$$

$$\tilde{R}_{\substack{m,n,\mu,\nu\\m-1,n-1,\mu,\nu}} = Pmn \quad , \quad \tilde{R}_{\substack{m,n,\mu,\nu\\m,n,\mu-1,\nu-1}} = P\mu\nu\,, \tag{31}$$

$$\tilde{R}_{\substack{m,n,\mu,\nu\\m+1,n+1,\mu,\nu}} = iU(m-n)\,, \quad \tilde{R}_{\substack{m,n,\mu,\nu\\m,n,\mu+1,\nu+1}} = iU(\mu-\nu)\,, \tag{32}$$

$$\tilde{R}_{\substack{m,n,\mu,\nu\\m-1,n,\mu,\nu}} = i\Omega m \quad , \quad \tilde{R}_{\substack{m,n,\mu,\nu\\m,n,\mu-1,\nu}} = i\Omega\mu\,, \tag{33}$$

$$\tilde{R}_{\substack{m,n,\mu,\nu\\m,n-1,\mu,\nu}} = -i\Omega n \quad , \quad \tilde{R}_{\substack{m,n,\mu,\nu\\m,n,\mu,\nu-1}} = -i\Omega\nu\,, \tag{34}$$

$$\tilde{R}_{\substack{m,n,\mu,\nu\\m-1,n,\mu+1,\nu}} = iJm \quad , \quad \tilde{R}_{\substack{m,n,\mu,\nu\\m+1,n,\mu-1,\nu}} = iJ\mu\,, \tag{35}$$

$$\tilde{R}_{\substack{m,n,\mu,\nu\\m,n-1,\mu,\nu+1}} = -iJn \quad , \quad \tilde{R}_{\substack{m,n,\mu,\nu\\m,n+1,\mu,\nu-1}} = -iJ\nu\,, \tag{36}$$

and zero everywhere else. The vector $\tilde{I}$ is constructed from the elements that provide an independent term for the equations, that is, $\tilde{R}_{\substack{m,n,\mu,\nu\\0,0,0,0}}$, while the matrix $\tilde{M}$ corresponds to all other elements.

References

[1] J.M. Raimond, R. Brune, and S. Haroche, Rev. Mod. Phys. **73**, 565 (2001).
[2] S.M. Girvin, M.H. Devoret, and R.J. Schoelkopf, Phys. Scr. **T137**, 014012 (2009).
[3] M.J. Hartmann, F.G.S.L. Brandão, and M.B. Plenio, Nature Phys. **2**, 849 (2006).
[4] D.G. Angelakis, M.F. Santos, and S. Bose, Phys. Rev. A **76**, 031805(R) (2007).
[5] A.D. Greentree, C. Tahan, J.H. Cole, and L.C.L. Hollenberg, Nature Phys. **2**, 856 (2006).
[6] D. E. Chang, V. Gritsev, G. Morigi, V. Vuletic, M. D. Lukin, and E. A. Demler, Nature Phys. **4**, 884 (2008).
[7] M. Kiffner and M.J. Hartmann, Phys. Rev. A **81**, 021806(R) (2010).
[8] T. Peyronel, O. Firstenberg, Q.-Y. Liang, S. Hofferberth, A.V. Gorshkov, T. Pohl, M.D. Lukin and V. Vuletić, Nature **488**, 57 (2012).
[9] S. Ritter, C. Nölleke, C. Hahn, A. Reiserer, A. Neuzner, M. Uphoff, M. Mücke, E. Figueroa, J. Bochmann, and G. Rempe, Nature **484**, 195 (2012).
[10] A.A. Houck, H.E. Tureci, and J. Koch, Nature Phys. **8**, 292 (2012).
[11] E. Lucero, R. Barends, Y. Chen, J. Kelly, M. Mariantoni, A. Megrant, P. O'Malley, D. Sank, A. Vainsencher, J. Wenner, T. White, Y. Yin, A.N. Cleland, and J.M. Martinis, Nature Phys. **8**, 719 (2012).
[12] M.J. Hartmann, F.G.S.L. Brandão, and M.B. Plenio, Laser & Photon. Rev. **2**, 527 (2008).
[13] A. Tomadin and R. Fazio, J. Opt. Soc. Am. **27**, A130 (2010).
[14] I. Carusotto, D. Gerace, H. Tureci, S. De Liberato, C. Ciuti, and A. Imamoğlu, Phys. Rev. Lett. **103**, 033601 (2009).
[15] F. Nissen, S. Schmidt, M. Biondi, G. Blatter, H.E. Türeci, and J. Keeling, Phys. Rev. Lett. **108**, 233603 (2012).
[16] M.J. Hartmann, Phys. Rev. Lett. **104**, 113601 (2010).
[17] J. Jin, D. Rossini, R. Fazio, M. Leib, and M. J. Hartmann, Phys. Rev. Lett. (to be published), arXiv:1302.2242
[18] E. del Valle, *Microcavity Quantum Electrodynamics* (VDM Verlag, 2010)

[19] E. del Valle, A. Gonzalez-Tudela, F. P. Laussy, C. Tejedor, and M. J. Hartmann, Phys. Rev. Lett. **109**, 183601 (2012).